Fisher Information and Delta Method for the Toothpaste Cap Problem

Setup

For the *toothpaste cap problem*, we have

$$\ell(\pi) = \log L(\pi) = k \log \pi + (N - k) \log(1 - \pi)$$

and

$$\hat{\pi} = \frac{1}{N} \sum_{i=1}^{N} y_i = \text{avg}(y) = \frac{k}{N}.$$

To estimate the variance, we need

$$\widehat{\text{Var}}(\hat{\pi}) \;\approx\; \left[\mathcal{I}_{\text{obs}}(\hat{\pi})\right]^{-1} = \left[- \left. \frac{d^2 \ell(\pi)}{d\pi^2} \right|_{\pi = \hat{\pi}} \right]^{-1}.$$

Finding $\widehat{\text{SE}}(\hat{\pi})$

Finding derivatives of ℓ

The first derivative is $\frac{d\ell}{d\pi} = \frac{k}{\pi} - \frac{N-k}{1-\pi}$.

The second derivative is $\frac{d^2\ell}{d\pi^2} = -\frac{k}{\pi^2} - \frac{N-k}{(1-\pi)^2}$ (see below).

Some tedious algebra

$$
\begin{aligned}
\frac{d\ell}{d\pi} &= \frac{k}{\pi} - \frac{N-k}{1-\pi}, \\
\frac{d^2\ell}{d\pi^2} &= \frac{d}{d\pi}\left(\frac{k}{\pi}\right) - \frac{d}{d\pi}\left(\frac{N-k}{1-\pi}\right) && \text{differentiate term by term} \\
&= k\,\frac{d}{d\pi}(\pi^{-1}) \;-\; (N-k)\,\frac{d}{d\pi}((1-\pi)^{-1}) && \text{rewrite as powers} \\
&= k(-1)\pi^{-2} \;-\; (N-k)\big[(-1)(1-\pi)^{-2}\cdot(-1)\big] && \text{power rule + chain rule} \\
&= -\frac{k}{\pi^2} \;-\; (N-k)(1-\pi)^{-2} \\
&= -\frac{k}{\pi^2} \;-\; \frac{N-k}{(1-\pi)^2}.
\end{aligned}
$$

Simplifying $-\ell''(\hat{\pi})$

Evaluate the negative second derivative at $\pi = \hat{\pi}$.

$$- \left.\frac{d^2\ell}{d\pi^2}\right|_{\pi=\hat{\pi}} = \frac{N}{\hat{\pi}(1-\hat{\pi})}$$

Thus,

$$\widehat{\text{Var}}(\hat{\pi}) \approx \left[- \left.\frac{d^2\ell}{d\pi^2}\right|_{\pi=\hat{\pi}} \right]^{-1} = \left[\frac{N}{\hat{\pi}(1-\hat{\pi})} \right]^{-1} = \frac{\hat{\pi}(1-\hat{\pi})}{N}$$

and

$$\boxed{\widehat{\text{SE}}(\hat{\pi}) \approx \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{N}}}$$

Some tedious algebra

Use the following identities:

- $k = N\hat{\pi}$. $\rightarrow$ Multiply both sides of $\hat{\pi} = k/N$ by N.
- $N - k = N(1-\hat{\pi})$. $\rightarrow$ $N - k = N - (N \cdot k/N) = N - N\hat{\pi} = N(1-\hat{\pi})$.
- $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$. $\rightarrow$ Common, useful identity.

$$\begin{aligned}
- \left.\frac{d^2\ell}{d\pi^2}\right|_{\pi=\hat{\pi}} &= \frac{k}{\hat{\pi}^2} + \frac{N-k}{(1-\hat{\pi})^2} && \text{evaluate at } \pi = \hat{\pi} \\
&= \frac{N\hat{\pi}}{\hat{\pi}^2} + \frac{N(1-\hat{\pi})}{(1-\hat{\pi})^2} && \text{use } k = N\hat{\pi},\ N-k = N(1-\hat{\pi}) \\
&= N\left(\frac{1}{\hat{\pi}} + \frac{1}{1-\hat{\pi}}\right) && \text{cancel a factor in each term} \\
&= N\left(\frac{\hat{\pi} + (1-\hat{\pi})}{\hat{\pi}(1-\hat{\pi})}\right) && \text{use } \tfrac{1}{a} + \tfrac{1}{b} = \tfrac{a+b}{ab} \\
&= \frac{N}{\hat{\pi}(1-\hat{\pi})} && \text{simplify}
\end{aligned}$$

Delta Method

We often report odds

$$\tau(\pi) = \frac{\pi}{1-\pi}, \qquad \widehat{\text{odds}} = \tau(\hat{\pi}) = \frac{\hat{\pi}}{1-\hat{\pi}}.$$

Finding the derivative of $\tau(\pi)$

For the delta method, we need the derivative of $\tau\pi$.

$$\tau'(\pi) = \frac{1}{(1-\pi)^2}.$$

> **i** Some tedious algebra
>
> $$\begin{aligned} \tau(\pi) &= \frac{\pi}{1-\pi} \\ \tau'(\pi) &= \frac{1\cdot(1-\pi) - \pi\cdot(-1)}{(1-\pi)^2} && \text{quotient rule} \\ &= \frac{1-\pi+\pi}{(1-\pi)^2} = \frac{1}{(1-\pi)^2}. && \text{combine terms} \end{aligned}$$

Plugging into the delta method equation

Recall that $\widehat{\mathrm{Var}}(\tau(\hat{\theta})) \;\approx\; \left(\tau'(\hat{\theta})\right)^2 \widehat{\mathrm{Var}}(\hat{\theta})$.

Applying this, we obtain

$$\widehat{\mathrm{Var}}(\widehat{\mathrm{odds}}) \approx \frac{\hat{\pi}}{N(1-\hat{\pi})^3}.$$

Taking the square root,

$$\boxed{\widehat{\mathrm{SE}}(\widehat{\mathrm{odds}}) = \sqrt{\tfrac{\hat{\pi}}{N(1-\hat{\pi})^3}}}.$$

Some tedious algebra

$$\begin{aligned}
\widehat{\mathrm{Var}}(\widehat{\mathrm{odds}}) &\approx \left(\tau'(\hat{\pi})\right)^2 \widehat{\mathrm{Var}}(\hat{\pi}) && \text{delta method} \\
&= \left(\frac{1}{(1-\hat{\pi})^2}\right)^2 \cdot \frac{\hat{\pi}(1-\hat{\pi})}{N} && \text{use } \ \tau'(\pi) = \frac{1}{(1-\pi)^2}, \quad \widehat{\mathrm{Var}}(\hat{\pi}) = \frac{\hat{\pi}(1-\hat{\pi})}{N} \\
&= \frac{1}{(1-\hat{\pi})^4} \cdot \frac{\hat{\pi}(1-\hat{\pi})}{N} && \text{square the derivative} \\
&= \frac{\hat{\pi}}{N(1-\hat{\pi})^3} && \text{cancel one factor of } (1-\hat{\pi}) \text{ and rearrange}
\end{aligned}$$