Week 1 Exercises

I created some notes that review the concepts required for these exercises, but you should draw on familiar textbooks and notes where possible.

And this is an ambitious set of exercises. Make a serious effort to freshen your mathematical toolkit. Your effort will make for a smoother semester. If you find yourself struggling to finish all the problems, then focus on 4(f), 4(g), 5, 7, 8, 9, 10, 13(1), 14, 15, 20, 21, 22, 24(c), 24(g), 25(b), 25(e), 26(c), and 27. Use the other exercises as needed to brush up on weaknesses.

1 Fractions, Logarithms, and Exponents

Exercise 1 Some Practice with Fractions

Simplify each of the following:

a. $\frac{x}{y} + \frac{z}{y}$

b. $\frac{m}{n} - \frac{r}{n}$

c. $\frac{p+q}{r} + \frac{s}{r}$

d. $\frac{2a}{b} + \frac{3a}{b}$

e. $\frac{x}{2} + \frac{y}{3}$

f. $\frac{1}{a} - \frac{1}{b}$ (write as a single fraction)

Exercise 2 Some Practice with Logarithms

You can assume arguments to logs are always positive, for the sake of these exercises.

Simplify each of the following:

a. $\log_{10}(10^5)$

b. $\log(e^3)$ (note: $\log(x)$ refers to the natural log in our class)

c. $\log(3x) + \log(4)$

d. $\log(x^3y^2)$

e. $\log\left(\frac{x^2}{y^3}\right)$

f. $\log\left(\prod_{i=1}^{n} x_i^2\right)$

Exercise 3 Some Practice with Exponents

Simplify each of the following:

a. $e^2 \cdot e^5$

b. $\left(e^3\right)^2$

c. $\frac{e^x}{e^4}$

d. $\left(\frac{1}{e}\right)^x$

e. $10^x \cdot 10^{-x}$

f. $\left(ab^2\right)^3$

Exercise 4 Some Practice Combining Fractions, Logarithms, and Exponents

Simplify each of the following:

a. $\dfrac{x^2+x}{x}$

b. $\dfrac{1}{2}\log(x^2)$

c. $\frac{e^{x+1}}{e}$

d. $\log\left(\dfrac{e^2}{x}\right)$

e. $\dfrac{\log(x)+\log(y)}{2}$

f. $\log\left(\dfrac{a^3b^2}{c}\right)$

g. $\log\left(\prod_{i=1}^{n}\pi^{y_i}(1-\pi)^{(1-y_i)}\right)$

Exercise 5 Inverse Logit

Show that the two common forms of the inverse-logit function

$$f_1(x) = \frac{1}{1+e^{-x}} \quad \text{and} \quad f_2(x) = \frac{e^x}{1+e^x}$$

are algebraically equivalent for every real number x.

2 Derivatives

Exercise 6 Some Practice with Derivatives

Differentiate each of the following:

a. Let $f(x)=5$

b. Let $f(x)=x^4$

c. Let $f(x)=3x^2-4x+7$

d. Let $f(x)=e^{2x}$

e. Let $g(t)=\log(t^3)$

f. Let $h(y) = y^3 \log(y)$

g. Let $f(x) = (x^2 + 1)^5$

h. Let $f(x) = \dfrac{e^x}{x}$

i. Let $q(u) = u^2 e^{u^3}$

j. Let $r(z) = \dfrac{\log(z^2 + 1)}{e^z}$

Exercise 7 An Important Preview

Let $\ell(\pi) = S\log(\pi) + (n - S)\log(1 - \pi)$ for $0 < \pi < 1$, where S and n (with $0 \leq S \leq n$) are fixed numerical constants. Find $\frac{d\ell(\pi)}{d\pi}$ and $\frac{d^2\ell(\pi)}{d\pi^2}$.

Exercise 8 Inverse Logit

Define $p(\theta) = \frac{1}{1+e^{-\theta}}$ for $\theta \in \mathbb{R}$. Find $p'(\theta)$ and $p''(\theta)$.

Exercise 9 Brambor, Clark, and Golder (2006)

Let $E[y \mid x_1, x_2] = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$, where $\beta_0, \dots, \beta_3$ are constants. Find the marginal effect of x_1 on y. That is, compute $\dfrac{\partial E[y \mid x_1, x_2]}{\partial x_1}$.

Exercise 10 Brambor, Clark, and Golder (2006) for Cubic Polynomial

Let $E[y \mid x] = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$, where $\beta_0, \dots, \beta_3$ are constants. Find the marginal effect of x on y. That is, compute $\dfrac{\partial E[y \mid x]}{\partial x}$.

3 Integrals

Exercise 11 Some Practice Problems for Integrals

Compute each integral.

a. $\int 7\,dx$

b. $\int_0^2 x^6\,dx$

c. $\int (3x^2 - 4)\,dx$

d. $\int_0^1 e^{3x}\,dx$

e. $\int \frac{1}{x}\,dx$

f. $\int 4x\,e^{x^2}\,dx$ *Hint:* let $u = x^2$.

g. $\int x\,e^x\,dx$ *Hint:* integration by parts with $u = x$.

h. $\int_0^1 x^2\,dx$

4 Matrices

Exercise 12 Some Practice with Transposes

Find the transpose of each of the following matrices:

a. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

b. $B = \begin{bmatrix} 0 & -1 & 2 \\ 5 & 3 & 1 \end{bmatrix}$

c. $C = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

Exercise 13 Some Practice with Matrix Multiplication

Problem 1

Multiply A and B using paper-and-pencil. Check your work with R.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}.$$

Problem 2

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \end{bmatrix}.$$

Compute $A'A$ (i.e., $A^\top A$) using paper-and-pencil. Check your work with R.

Exercise 14 Matrices, OLS, and R

For the linear model $y = X\beta + \varepsilon$, where X is an $n \times k$ full-rank design matrix (including a column of ones), we define

- OLS estimator: $\widehat{\beta} = (X'X)^{-1}X'y$.
- Residual vector: $e = y - X\widehat{\beta}$.
- Predicted values: $\hat{y} = X\widehat{\beta}$.
- Classical variance–covariance matrix: $\text{Var}_{\text{cl}}(\widehat{\beta}) = \hat{\sigma}^2 (X'X)^{-1}$, where $\hat{\sigma}^2 = \dfrac{e'e}{n-k}$.

1. Write an R function `ols(X, y)` that returns a list with the coefficient estimates `beta_hat`, the classical variance matrix `Var_cl`, the residuals `e`, and the predicted values `y_hat`.
2. Use the **`penguins`** dataset in the {palmerpenguins} package to test your function. The following starter code prepares X and y.

```
# load packages
library(palmerpenguins)

# 1. drop NAs
penguins_complete <- na.omit(penguins)

# 2. make response vector (y)
y <- penguins_complete$body_mass_g # numeric vector (n × 1)

## 3. make design matrix (X)
X_predictors <- as.matrix(penguins_complete[ ,
                  c("bill_length_mm",
                    "bill_depth_mm",
```

```
                          "flipper_length_mm")])

X <- cbind("(Intercept)" = 1, X_predictors)
```

Exercise 15 Gradients and Hessians

Consider the function of two variables $f(x, y) = 3x^2y + 2e^{xy} - y^3$. Compute (1) the gradient vector $\nabla f(x, y)$ and (2) the Hessian matrix $H_f(x, y)$.

5 Probability Theory

Exercise 16 Some Results of the Axioms and Definition of Probability

Prove the following results:

a. $\Pr(\emptyset) = 0$.
b. If event $A \subseteq B$, then $\Pr(A) \leq \Pr(B)$.
c. For event A, $0 \leq \Pr(A) \leq 1$.
d. For any event A, $\Pr(A^c) = 1 - \Pr(A)$.

Hints

a. Use Axiom 3.
b. Notice that $B = A \cup (B \cap A^c)$. Then use the Additional Rule for Two Disjoint Events.
c. Axiom 1 establishes that $0 \leq \Pr(A)$. Now show that $\Pr(A) \leq 1$. To do this, the result from (b).
d. Notice that $\Pr(S) = \Pr(A) + \Pr(A^c)$ and follow this forward.

Exercise 17 Simplifying the Multiplication Rule

Simplify $\Pr(A \mid B)$ for the following scenarios.

a. $A \subset B$ and $\Pr(B) > 0$.
b. A and B are disjoint and $\Pr(B) > 0$.
c. B is the empty set (tricky!).
d. B is the sample space S.

Exercise 18

Suppose A and B are independent and $\Pr(B) < 1$. Find $\Pr(A^c|B^c)$ in terms of A and B. Prove that A^c and B^c are independent.

Exercise 19 Independence when $\Pr(B) = 0$

Suppose A and B are events and $\Pr(B) = 0$. (A is any event.) Find $\Pr(A \cap B)$. Prove that A and B are independent.

Exercise 20 Sixes

Suppose a six-sided die is rolled 10 times. What's the probability of...

a. all sixes?
b. not all-sixes?
c. all not-sixes?

Exercise 21 Drug Testing (Bayes' Rule)

A drug test is used to detect the presence of a banned substance in professional athletes. Suppose that 2 in every 1,000 athletes use the substance. The test correctly identifies users 98% of the time. However, it also produces a false positive 1% of the time for non-users. You are randomly selected for testing, and your result comes back positive. Use Bayes' rule to compute the probability that you actually use the substance.

Exercise 22 Bernoulli Distribution

Let $X \sim \text{Bernoulli}(p)$ with pmf

$$f(x;p) = \begin{cases} p, & x = 1, \\ 1-p, & x = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad 0 < p < 1.$$

Compute

1. the cumulative distribution function $F(x) = \Pr(X \leq x)$,

2. the expected value $\mathrm{E}[X]$, and

3. the variance $\mathrm{Var}(X)$.

Exercise 23   Exponential Distribution (Challenging)

Note: This is a very challenging exercise that requires you to evaluate an increasingly difficult set of integrals. It's acceptable to use a symbolic solver for these questions, but it's also good to see what "simple" integrals can look like in "simple" real problems.

Let $X \sim \mathrm{Exponential}(\lambda)$ with probability density function

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \qquad \lambda > 0.$$

Compute

1. the cumulative distribution function $F(x) = \Pr(X \leq x)$,

2. the expected value $\mathrm{E}[X]$, and

3. the variance $\mathrm{Var}(X)$.

Hints:

- Use integration by parts to compute $\mathrm{E}[X]$. Set $u = x$ and $dv = \lambda e^{-\lambda x}\,dx$.

- To compute $\mathrm{E}[X^2]$, apply integration by parts **twice**. First let $u = x^2$ and $dv = \lambda e^{-\lambda x} dx$. This will leave an integral of the form $\int x e^{-\lambda x}\,dx$, which you already solved when computing $\mathrm{E}[X]$.
- Use the formula $\mathrm{Var}(X) = \mathrm{E}[X^2] - \left(\mathrm{E}[X]\right)^2$.

Exercise 24   Some Practice with Expectations

Use the stated rules to simplify each expression.

a. $E[\,2X + 3Y\,]$

b. $E[\,5X - 7Y\,]$

c. $E[\,4X + Y - 2Z\,]$

d. $E[\,10\,]$

e. $E[\,\pi\,]$

f. $E[\,2XY\,]$ (assume X and Y independent)

g. Let X take values $1, 2, 3$ each with probability $1/3$. Compute $E[X^2]$.

h. Let X have pdf $f_X(x) = 2x$ on $[0, 1]$. Compute $E[X^3]$.

i. Let X take values $0, 1$ with $P(X = 1) = p$. Compute $E[\log(1 + X)]$.

Exercise 25 Some Practice with Variances

Use the stated rules to simplify each expression.

a. $\text{Var}(X)$ in terms of $E[X^2]$ and $E[X]$

b. $\text{Var}(3X)$ (express in terms of $E[X^2]$ and $E[X]$)

c. $\text{Var}(X + Y)$ in terms of $\text{Var}(X)$, $\text{Var}(Y)$, and $\text{Cov}(X, Y)$

d. $\text{Var}(X + Y)$ if X and Y are independent

e. $\text{Var}(2X - 3Y)$ (assume independence)

Exercise 26 Some Practice with Covariances

Use the stated rules to simplify each expression.

a. $\text{Cov}(2X, 3Y)$

b. $\text{Cov}(-X, 4Y)$

c. $\text{Cov}(2X - 3Y, Z)$

d. $\text{Cov}(2X, Y)$ if X and Y are independent

Exercise 27 OLS, Interactions, and R

A Theoretical Result

For this portion, it will be helpful to have Brambor, Clark, and Golder (2006) handy.

Consider the linear interaction model $Y = \beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ + \varepsilon$ estimated by OLS. For a given value $Z = z$, the marginal effect of X on $E(Y)$ is $\beta_1 + \beta_3 z$, with plug-in estimate $\widehat{\text{ME}}_X(z) = \hat{\beta}_1 + \hat{\beta}_3 z$ (i.e., see eq. 13 in Brambor, Clark, and Golder 2006).

Let $\widehat{\text{Var}}(\hat{\beta}_1)$, $\widehat{\text{Var}}(\hat{\beta}_3)$, and $\widehat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_3)$ denote the corresponding elements from the estimated variance–covariance matrix of $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)$. Treat z as fixed. Re-derive the standard error of $\widehat{\text{ME}}_X(z)$ shown in eq. 8 of Brambor, Clark, and Golder (2006). In other words, show that

$$\widehat{\text{SE}}\{\widehat{\text{ME}}_X(z)\} = \sqrt{\widehat{\text{Var}}\left(\hat{\beta}_1 + \hat{\beta}_3 z\right)} = \sqrt{\widehat{\text{Var}}(\hat{\beta}_1) + z^2\,\widehat{\text{Var}}(\hat{\beta}_3) + 2z\,\widehat{\text{Cov}}(\hat{\beta}_1, \hat{\beta}_3)}\,.$$

A Computational Result

For this portion, it will be helpful to have Clark and Golder (2006) handy.

The code below reproduces Clark and Golder's (2006) 1946-2000 Established Democracies model in Table 2 on p. 698. See also eq. 4 on p. 695 for their model specification.

```
# load packages
library(sandwich)  # for robust SEs

# load Clark and Golder's data
cg <- crdata::cg2006  # from my data package

# reproduce regression
f <- enep ~ eneg*log(average_magnitude) + eneg*upper_tier + en_pres*proximity
fit <- lm(f, data = cg)

# grab estimated coefs and var matrix
beta_hat <- coef(fit)
V_hat <- vcovCL(fit, cluster = ~ country, type = "HC1")

# print table (shows we reproduce!)
# modelsummary::modelsummary(fit, vcov = V_hat, fmt = 2)
```

1. Derive the formula for the marginal effect of `eneg` and its SE. *Hint: It depends on both `average_magnitude` and `upper_tier`.
2. Use R to compute the marginal effect of `eneg` for the maximum value of `log(average_magnitude)` and it's SE. (You can fix `upper_tier` to 0, which simplifies the calculation.)

3. Use R to compute the marginal effect of `eneg` and SE for a *range* of values from the minimum value of `log(average_magnitude)`. Compute a 90% confidence interval and create a marginal effect plot like those in Figure 1 of Clark and Golder (2006). *Note: you should have the log of `average_magnitude` (not `average_magnitude` itself) along the x-axis.*